EXAMPLE 6 Using the Rational Power Rule

(a) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

Notice that $\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}(x^{-1/3}) = \dfrac{2}{3x^{1/3}}$

The original function is defined for all real numbers, but the derivative is undefined at $x = 0$. Recall Figure 3.12, which showed that this function's graph has a *cusp* at $x = 0$.

(c)
$$\begin{aligned}\frac{d}{dx}(\cos x)^{-1/5} &= -\frac{1}{5}(\cos x)^{-6/5} \cdot \frac{d}{dx}(\cos x)\\ &= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)\\ &= \frac{1}{5}\sin x(\cos x)^{-6/5}\end{aligned}$$

Now try Exercise 33.

2. $y_1 = \frac{2}{3}\sqrt{9-x^2},\ y_2 = -\frac{2}{3}\sqrt{9-x^2}$ **5.** $y_1 = \sqrt{2x+3-x^2},\ y_2 = -\sqrt{2x+3-x^2}$ **8.** $y' = \dfrac{xy^2}{x^2-y+x}$

Quick Review 3.7

In Exercises 1–5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$ $y_1 = \sqrt{x},\ y_2 = -\sqrt{x}$

2. $4x^2 + 9y^2 = 36$

3. $x^2 - 4y^2 = 0$ $y_1 = \dfrac{x}{2},\ y_2 = -\dfrac{x}{2}$

4. $x^2 + y^2 = 9$ $y_1 = \sqrt{9-x^2},\ y_2 = -\sqrt{9-x^2}$

5. $x^2 + y^2 = 2x + 3$

In Exercises 6–8, solve for y' in terms of y and x.

6. $x^2y' - 2xy = 4x - y$ $y' = \dfrac{4x - y + 2xy}{x^2}$

7. $y'\sin x - x\cos x = xy' + y$ $y' = \dfrac{y + x\cos x}{\sin x - x}$

8. $x(y^2 - y') = y'(x^2 - y)$

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$ $x^{3/2} - x^{5/6}$

10. $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$ $x^{-1/2} + x^{-5/6}$

25. (a) $y = 2\pi x - 2\pi$ **(b)** $y = -\dfrac{x}{2\pi} + \dfrac{1}{2\pi}$

Section 3.7 Exercises

In Exercises 1–8, find dy/dx.

4. $\dfrac{y}{x} - (x+y)^2$ or $\dfrac{1 - 3x^2 - 2xy}{x^2 + 1}$

1. $x^2y + xy^2 = 6$ $-\dfrac{2xy + y^2}{2xy + x^2}$

2. $x^3 + y^3 = 18xy$ $\dfrac{6y - x^2}{y^2 - 6x}$

3. $y^2 = \dfrac{x-1}{x+1}$ $\dfrac{1}{y(x+1)^2}$

4. $x^2 = \dfrac{x-y}{x+y}$

5. $x = \tan y$ $\cos^2 y$

6. $x = \sin y$ $\sec y$

7. $x + \tan(xy) = 0$ See page 164.

8. $x + \sin y = xy$ $\dfrac{1-y}{x - \cos y}$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13,\quad (-2, 3)$ $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $2/3$

10. $x^2 + y^2 = 9,\quad (0, 3)$ $\dfrac{dy}{dx} = -\dfrac{x}{y}$, 0

11. $(x-1)^2 + (y-1)^2 = 13,\quad (3, 4)$ $\dfrac{dy}{dx} = -\dfrac{x-1}{y-1}$, $-2/3$

12. $(x+2)^2 + (y+3)^2 = 25,\quad (1, -7)$ See page 164.

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$ See page 164.

14. $x = \cos y$ See page 164.

15. $x^3 + y^3 = xy$ See page 164.

16. $x^2 + 4xy + 4y^2 - 3x = 6$ See page 164.

In Exercises 17–26, find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1,\quad (2, 3)$ **(a)** $y = \frac{7}{4}x - \frac{1}{2}$ **(b)** $y = -\frac{4}{7}x + \frac{29}{7}$

18. $x^2 + y^2 = 25,\quad (3, -4)$ **(a)** $y = \frac{3}{4}x - \frac{25}{4}$ **(b)** $y = -\frac{4}{3}x$

19. $x^2y^2 = 9,\quad (-1, 3)$ See page 164.

20. $y^2 - 2x - 4y - 1 = 0,\quad (-2, 1)$ **(a)** $y = -x - 1$ **(b)** $y = x + 3$

21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0,\quad (-1, 0)$ See page 164.

22. $x^2 - \sqrt{3}xy + 2y^2 = 5,\quad (\sqrt{3}, 2)$ **(a)** $y = 2$ **(b)** $y = \sqrt{3}$

23. $2xy + \pi\sin y = 2\pi,\quad (1, \pi/2)$ See page 164.

24. $x\sin 2y = y\cos 2x,\quad (\pi/4, \pi/2)$ **(a)** $y = 2x$ **(b)** $y = -\frac{1}{2}x + \frac{5\pi}{8}$

25. $y = 2\sin(\pi x - y),\quad (1, 0)$

26. $x^2\cos^2 y - \sin y = 0,\quad (0, \pi)$ **(a)** $y = \pi$ **(b)** $x = 0$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$ See page 164.

28. $x^{2/3} + y^{2/3} = 1$ See page 164.

29. $y^2 = x^2 + 2x$ See page 164.

30. $y^2 + 2y = 2x + 1$ See page 164.

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$ $(9/4)x^{5/4}$

32. $y = x^{-3/5}$ $(-3/5)x^{-8/5}$

33. $y = \sqrt[3]{x}$ $(1/3)x^{-2/3}$

34. $y = \sqrt[4]{x}$ $(1/4)x^{-3/4}$

35. $y = (2x+5)^{-1/2}$ $-(2x+5)^{-3/2}$

36. $y = (1 - 6x)^{2/3}$ $-4(1-6x)^{-1/3}$

37. $y = x\sqrt{x^2+1}$ $x^2(x^2+1)^{-1/2} + (x^2+1)^{1/2}$

38. $y = \dfrac{x}{\sqrt{x^2+1}}$ $(x^2+1)^{-3/2}$

39. $y = \sqrt{1 - \sqrt{x}}$ See page 164.

40. $y = 3(2x^{-1/2} + 1)^{-1/3}$ See page 164.

41. $y = 3(\csc x)^{3/2}$ See page 164.

42. $y = [\sin(x+5)]^{5/4}$ See page 164.

43. Which of the following could be true if $f''(x) = x^{-1/3}$? (b), (c), and (d)

(a) $f(x) = \frac{3}{2}x^{2/3} - 3$ **(b)** $f(x) = \frac{9}{10}x^{5/3} - 7$

(c) $f'''(x) = -\frac{1}{3}x^{-4/3}$ **(d)** $f'(x) = \frac{3}{2}x^{2/3} + 6$

44. Which of the following could be true if $g''(t) = 1/t^{3/4}$? (a) and (c)

(a) $g'(t) = 4\sqrt[4]{t} - 4$ **(b)** $g'''(t) = -4/\sqrt[4]{t}$

(c) $g(t) = t - 7 + (16/5)t^{5/4}$ **(d)** $g'(t) = (1/4)t^{1/4}$

45. ***The Eight Curve*** **(a)** Find the slopes of the figure-eight-shaped curve

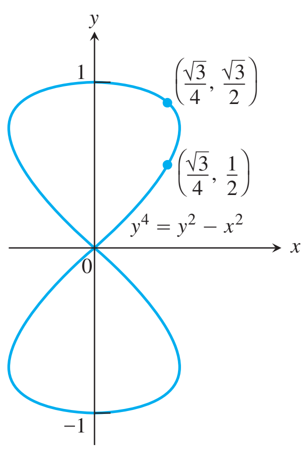
$$y^4 = y^2 - x^2$$

at the two points shown on the graph that follows.

(b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{t^2 - t^4}, \quad y_1(t) = t,$$
$$x_2(t) = -\sqrt{t^2 - t^4}, \quad y_2(t) = t,$$

to graph the curve. Specify a window and a parameter interval.



46. ***The Cissoid of Diocles (dates from about 200 B.C.)***

(a) Find equations for the tangent and normal to the cissoid of Diocles,

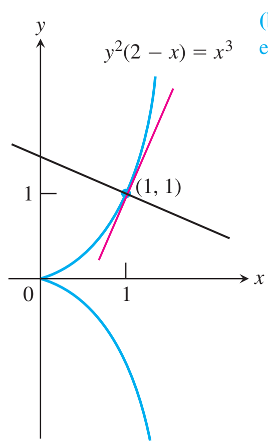
$$y^2(2 - x) = x^3,$$

at the point (1, 1) as pictured below.

(b) Explain how to reproduce the graph on a grapher.

(a) Tangent: $y = 2x - 1$

normal: $y = -\frac{1}{2}x + \frac{3}{2}$



(b) One way is to graph the equations

$$y = \pm\sqrt{\frac{x^3}{2 - x}}.$$

47. **(a)** Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$. $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal: -1.

(b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$. The slope is 3/2.

48. Grouping Activity

(a) Show that the relation There are three values: 1, $\frac{-1 \pm \sqrt{5}}{2}$

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

(b) On a small enough square with center (2, 1), the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$. $f'(2) = 1$, $f''(2) = -4$

49. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

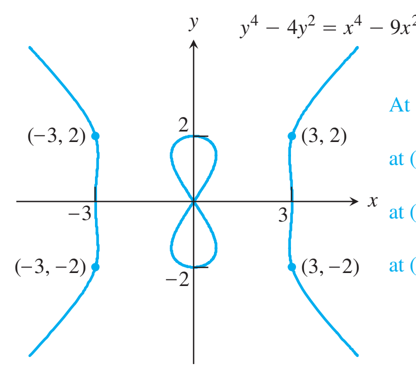
50. Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

51. ***Orthogonal Curves*** Two curves are *orthogonal* at a point of intersection if their tangents at that point cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and (1, −1). Use parametric mode to draw the curves and to show the tangent lines.

52. The position of a body moving along a coordinate line at time t is $s = (4 + 6t)^{3/2}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 2$ sec.

53. The velocity of a falling body is $v = 8\sqrt{s - t} + 1$ feet per second at the instant t(sec) the body has fallen s feet from its starting point. Show that the body's acceleration is 32 ft/sec².

54. ***The Devil's Curve (Gabriel Cramer [the Cramer of Cramer's Rule], 1750)*** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



At (3, 2): $\frac{27}{8}$; at (−3, 2): $-\frac{27}{8}$; at (−3, −2): $\frac{27}{8}$; at (−3, −2): $-\frac{27}{8}$

55. ***The Folium of Descartes*** (See Figure 3.47 on page 157)

(a) Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4). **(a)** At (4, 2): $\frac{5}{4}$; at (2, 4): $\frac{4}{5}$

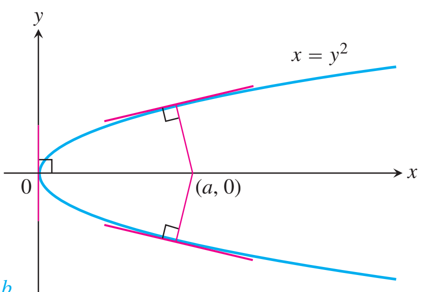
(b) At what point other than the origin does the folium have a horizontal tangent? At $(3\sqrt[3]{2}, 3\sqrt[3]{4}) \approx (3.780, 4.762)$

(c) Find the coordinates of the point A in Figure 3.47, where the folium has a vertical tangent. At $(3\sqrt[3]{4}, 3\sqrt[3]{2}) \approx (4.762, 3.780)$

56. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point? $(3, -1)$

57. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$. At $(-1, -1)$: $y = -2x - 3$; at $(3, -3)$: $y = -2x + 3$

58. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



The normal at the point (b^2, b) is: $y = -2bx + 2b^3 + b$.
This line intersects the x-axis at $x = b^2 + \frac{1}{2}$, which must be greater than $\frac{1}{2}$ if $b \neq 0$.
The two normals are perpendicular when $a = 3/4$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False The slope of $xy^2 + x = 1$ at $(1/2, 1)$ is 2. Justify your answer. False. It is equal to -2.

60. True or False The derivative of $y = \sqrt[3]{x}$ is $\frac{1}{3x^{2/3}}$. Justify your answer. True. By the power rule.

In Exercises 61 and 62, use the curve $x^2 - xy + y^2 = 1$.

61. Multiple Choice Which of the following is equal to dy/dx? A

(A) $\frac{y - 2x}{2y - x}$ (B) $\frac{y + 2x}{2y - x}$

(C) $\frac{2x}{x - 2y}$ (D) $\frac{2x + y}{x - 2y}$

(E) $\frac{y + 2x}{x}$

62. Multiple Choice Which of the following is equal to $\frac{d^2y}{dx^2}$? A

(A) $-\frac{6}{(2y - x)^3}$ (B) $\frac{10y^2 - 10x^2 - 10xy}{(2y - x)^3}$

(C) $\frac{8x^2 - 4xy + 8y^2}{(x - 2y)^3}$ (D) $\frac{10x^2 + 10y^2}{(x - 2y)^3}$

(E) $\frac{2}{x}$

63. Multiple Choice Which of the following is equal to dy/dx if $y = x^{3/4}$? E

(A) $\frac{3x^{1/3}}{4}$ (B) $\frac{4x^{1/4}}{3}$ (C) $\frac{3x^{1/4}}{4}$ (D) $\frac{4}{3x^{1/4}}$ (E) $\frac{3}{4x^{1/4}}$

64. Multiple Choice Which of the following is equal to the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$? C

(A) $-\frac{1}{\sqrt{2}}$ (B) $-\sqrt{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$ (E) 0

Extending the Ideas

65. *Finding Tangents*

(a) Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) has equation

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

(b) Find an equation for the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1).

66. *End Behavior Model* Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that

(a) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$.

(b) $g(x) = (b/a)|x|$ is an end behavior model for

$$f(x) = (b/a)\sqrt{x^2 - a^2}.$$

(c) $g(x) = -(b/a)|x|$ is an end behavior model for

$$f(x) = -(b/a)\sqrt{x^2 - a^2}.$$

7. $-\frac{1}{x}\cos^2(xy) - \frac{y}{x}$ **12.** $\frac{dy}{dx} = -\frac{x + 2}{y + 3}$, $3/4$

13. $\frac{dy}{dx} = \frac{2xy - y^2}{2xy - x^2}$, defined at every point except where $x = 0$ or $y = x/2$

14. $\frac{dy}{dx} = -\frac{1}{\sin y}$, defined at every point except where $y = k\pi$, k any integer

15. $\frac{dy}{dx} = \frac{3x^2 - y}{x - 3y^2}$, defined at every point except where $y^2 = x/3$

16. $\frac{dy}{dx} = \frac{3 - 2x - 4y}{4x + 8y}$, defined at every point except where $y = -\frac{1}{2}x$

19. (a) $y = 3x + 6$ **(b)** $y = -\frac{1}{3}x + \frac{8}{3}$

21. (a) $y = \frac{6}{7}x + \frac{6}{7}$ **(b)** $y = -\frac{7}{6}x - \frac{7}{6}$

23. (a) $y = -\frac{\pi}{2}x + \pi$ **(b)** $y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

27. $\frac{dy}{dx} = -\frac{x}{y}$

$\frac{d^2y}{dx^2} = -\frac{(x^2 + y^2)}{y^3} = -\frac{1}{y^3}$

28. $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

$\frac{d^2y}{dx^2} = \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}} = \frac{1}{3x^{4/3}y^{1/3}}$

29. $\frac{dy}{dx} = \frac{x + 1}{y}$

$\frac{d^2y}{dx^2} = \frac{y^2 - (x + 1)^2}{y^3} = -\frac{1}{y^3}$

30. $\frac{dy}{dx} = \frac{1}{y + 1}$

$\frac{d^2y}{dx^2} = -\frac{1}{(y + 1)^3}$

39. $-\frac{1}{4}(1 - x^{1/2})^{-1/2}x^{-1/2}$

40. $x^{-3/2}(2x^{-1/2} + 1)^{-4/3}$

41. $-\frac{9}{2}(\csc x)^{3/2}\cot x$

42. $\frac{5}{4}[\sin(x + 5)]^{1/4}\cos(x + 5)$